THE UNIVERSITY OF ZAMBIA

SOCIOLOGY

A NOTE ON THE "L" TEST AND THE "L' " TEST AS USEFUL ALTERNATIVES TO THE χ^2 TEST IN THE ANALYSIS OF SMALL-SAMPLE DATA

Wim M.J. van Binsbergen December 1972

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Introduction

In social statistics, we may use the χ^2 test if we wish to analyze (typically in data derived from a small sample, and about the underlying distribution of which we do not want to make assumptions - so that we cannot use a parametric test):

- 1. Whether there exists a statistically significant difference between two or more sets of tabulated, empirical (stochastical) data. Typically:

	X	Y	X+Y
A	p	q	p+q
B	r	s	r+s
A+B	p+r	q+s	p+q+r+s

table 1

Example: We take an aselect sample of 100 UNZA students; the variables X and Y stand for, resp., male and female; A and B stand for, resp. baptized and non-baptized in any Christian church. Our table could have the following form:

	male	female	total
baptized	70	8	78
non-bapt.	19	3	22
total	89	11	100

table 2

The χ^2 test now yields a particular statistic (in this case: $\chi^2 = 0.20$), associated with a particular number of degrees of freedom (df = (a-1).(b-1) = 1, where a is number of row entries and b is number of column entries.)Consulting a χ^2 table we find that the probability of "finding the distribution as in table 2 whilst in reality there is no difference between male and female in their chance of being baptized," is between 50% and 70%; as this is much higher than any usual significance level (5%, 1% etc.), we conclude that there is no demonstrable difference between boys and girls in our sample in their chances of being baptized.

- 2. Whether any set of empirical, stochastical data deviates significantly from a standard expectation. Typically:

	A	B	A+B
observed	p	q	p+q
expected	r	p+q−r	p+q

table 3

Example: Suppose the national proportion of baptized as against non-baptized is 40%. Using the data from table 3, we find:

	baptized	non-bapt.	total
observed (UNZA boys)	70	19	89
expected	$\frac{40}{100} \cdot 89 = 35.6$	$\frac{60}{100} \cdot 89 = 53.4$	89

table 4

Again, χ^2 will yield a statistic which, under df=1, will tell us whether UNZA boys have significantly more chance of being baptized than is to be expected on the basis of the national average. (Here, $\chi^2 = 55.39$, which is significant at the 5%, 1% and even 0.1 % level.)

Disadvantages of the χ^2 test.

Although widely used, and discussed in nearly every introductory book on social statistics (e.g. Siegel, n.d., 42f, 104f, 175f), the χ^2 test has important disadvantages:

- its calculation is tedious and time-consuming;
- for very small samples the test requires a "correction for continuity";
- for any sample the test requires a minimum value per cell of at least 5 units. (Note that I sinned against this rule when computing χ^2 for table 2!)

Fortunately, another type of tests is available which, though little known (Woolf 1957; Spitz 1961; Spitz 1965) is very suitable for the research problems arising from small sample research in the social sciences (Binsbergen 1970). These tests were called by Spitz: l test (for two or more stochastic distributions) and l' test (for one stochastic distribution as compared to a standard expectation); the name derives from the extensive use of logarithms that is made in these tests. Once one is accustomed to the use of (natural) logarithms, calculations are extremely simple; none of the disadvantages of the χ^2 occur (there is no correction for continuity neither a minimal requirement per cell); and, once the statistic is calculated, it can be interpreted with a normal χ^2 table, observing the same rules about degrees of freedom as apply to χ^2. (1)

I will not discuss the mathematical background of the l and l' tests in relation to the χ^2 test (cf. Mood 1950; Woolf 1957; Spitz 1961), but shall only demonstrate the application of these tests on the basis of the (fictive) data used previously (tables 2 and 4).

The l test.

The general formula for the computation of l is:

$$l = 2\left[n \ln n + \sum_{i=1}^{a} \sum_{j=1}^{b} f_{ij} . \ln f_{ij} - \sum_{i=1}^{a} n_i . \ln n_i - \sum_{j=1}^{b} f_j . \ln f_j \right]$$

(where:

n	=	sample size
ln	=	natural logarith (basis e)
a	=	number of row entries
i	=	rank number of a particular row entry ($1 \geqslant i \geqslant a$)
b	=	number of column entries
j	=	rank number of a particular column entry($1 \geqslant j \geqslant b$)
f_{ij}	=	frequency in a particular, identified cell
n_i	=	bottom total of a particular column
f_j	=	right-hand total of a particular row)

This may look extremely complicated, but things become clearer if we rewrite table 1 accordingly (p.t.o.):

(1) Thus, for the l test, df = (a-1).(b-1); for the l' test, df = a - 1.

	X	Y	X+Y
A	f_{11}	f_{21}	f_1
B	f_{12}	f_{22}	f_2
A+B	n_1	n_2	n

table 5

Substituting now the empirical values as shown in table 2, the statistic is computed as follows:

$$1 = 2[(100.\ln 100)+(70.\ln 70 + 8.\ln 8 + 19.\ln 19 + 3.\ln 3) - (89.\ln 89 + 11.\ln 11) - (78.\ln 78 + 22.\ln 22)] = 0.198.$$

The various values of a.ln a can be calculated on the basis of a table of natural logarithms, but of course it saves much time if one uses a table of a.ln a straight-away; such a table is given by Spitz (1961; 1965).

So the standard procedure for the 1 test, no matter the number of row and column entries in our table, is:

1. add the values of a.ln a for, successively, n (= sample size) and <u>all</u> cell frequencies;
2. substract from this sum the values of a.ln a for all bottom and all right-hand totals;
3. multiply the result by two.

As you see, the statistic thus obtained has exactly the same value as χ^2 would have had; differences that may show are due to rounding-off.

The 1' test.

The general formula for the computation of 1' is:

$$1' = 2 \sum_{i=1}^{a} f_i.\ln\left(\frac{f_i}{e_i}\right)$$

(where: f_i = frequency observed in the i-th column entry
e_i = frequency expected in the i-th column entry
ln = natural logarithm
i = rank number of a particular column entry ($1 \geq i \geq a$)
a = number of column entries.)

Substituting the empirical and expected values as shown in table 4, we get:

$$1' = 2\left(70.\ln \frac{70}{(40/100).89} + 19.\ln \frac{19}{(60/100).89} \right) =$$

$$2\left(70.\ln \frac{700}{356} + \frac{190}{534} \right) =$$

(Note that $\ln \frac{a}{b} = \ln a - \ln b$)

$$2 \left[70.(\ln 700 - \ln 356) + 19.(\ln 190 - \ln 534) \right] = 55.4.$$

This is again the same result as χ^2 would have given. For the calculation of 1' a table of natural logarithms is obviously indispensable.

The standard procedure for the computation of 1' (no matter the number of column entries, i.e. of categories, in our data) is:

1. Calculate the expected value as a fraction of the total sample size;
2. per column entry, divide the observed value by the expected value, and simplify without writing the fraction as a decimal fraction (this because it is tedious to find the natural logarithms of decimal fractions);
3. per column entry, subtract the natural logarithm of the expected value from the natural logarithm of the observed value - and multiply the result by the observed value;
4. add these products for each column entry;
5. multiply the sum by 2.

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